DERIVED DEFORMATION THEORY AND KOSZUL DUALITY – PARTICIPANT TALKS

OWEN GWILLIAM AND CLAUDIA SCHEIMBAUER

Part of the course will be talks given by the participants which cover background material and motivation coming from more classical examples and outlook topics. We suggest the topics below, but further suggestions are welcome!

Part 1. First topics

1. Introduction to model categories (April 25)

The goal of this talk is to cover the main definitions and constructions of model categories needed throughout the course.

Definitions: model category, Quillen adjunctions, Quillen equivalences, derived functors
Constructions: homotopy category is equivalent to the category of fibrant-cofibrant objects modulo weak equivalence, transfer of model structures
Examples: sSet and Top plus their Quillen equivalence, chain complexes (a.k.a. differential graded vector spaces) over $\mathbb{Q}$ and transferred model structure to commutative dg algebras

References: Some standard texts on model categories (amongst others) are Dwyer-Spalinski – Homotopy theories and model categories, Hovey – Model Categories, Hirschhorn – Model categories and their localizations. You might also find Goerss-Schemmerhorn – Model Categories and Simplicial Methods useful.

2. The model category of differential graded Lie algebras (April 26)

In this talk differential graded Lie algebras are revisited in the perspective of model categories. At the end we should see that the natural Chevalley-Eilenberg cochain functor does not fit into the model category framework, motivating a more flexible framework for higher categories.

Reminder on definition of model category of differential graded vector spaces, transfer to dg Lie algebras and dg associative algebras, universal enveloping algebra as Quillen functor, triangle of Quillen adjunctions, Poincaré-Birkhoff-Witt, Chevalley-Eilenberg functors

References: The main reference is Lurie – DAG X: Formal Moduli Problems; it is also nicely explained in Mauro Porta's master thesis – Derived formal moduli problems available at `http://algant.eu/documents/theses/porta.pdf`

3. Quasi-categories (May 3)

This talk introduces quasi-categories (also called weak Kan complexes) as a model for $(\infty, 1)$-categories.

Definitions: quasi-category, Joyal model structure on sSet, functors and functor category, (co)limits, (adjunctions and presentability if time permits)
Constructions: nerve functors from (simplicially enriched) categories to quasi-categories, quasi-categories coming from model categories

References: The original reference is Joyal – Notes on quasi-categories available at `http://www.math.uchicago.edu/~may/IMA/Joyal.pdf` and Lurie – Higher Topos Theory: Chapter 1 – 2,
expository accounts are Groth – A short course on ∞-categories available at `http://arxiv.org/pdf/`

`1007.2925.pdf`, Omar Antolín Camarena – A Whirlwind Tour of the World of $(\infty, 1)$-categories available at `https://arxiv.org/pdf/1303.4669`

Part 2. Rational homotopy theory and L_∞-algebras

4. Quillen's approach to rational homotopy theory

In the first reference below Quillen constructed a sequence of Quillen equivalences between a model category of simply connected rational topological spaces and dg Lie algebras concentrated in positive degrees. The goal of the talk is to explain this construction and applications.

References: Quillen – Rational homotopy theory, Berglund – Lecture notes on Rational homotopy theory

5. Introduction to L_∞-algebras

Explain the notion of an L_∞-algebra and their morphisms. Discuss some examples from Kontsevich-Soibelman's book and explain their appearance in Hinich's theorem. Finally, use this language to state Kontsevich's formality theorem in deformation quantization.

References: Hinich – DG coalgebras as formal stacks, Kontsevich-Soibelman – Topics in algebra. Deformation theory, Cattaneo - Indelicato – Formality and Star Products available at `http://arxiv.org/abs/math/0403135`

6. Homotopy transfer and L_∞-algebras

The homotopy transfer theorem gives a general procedure to transfer an algebraic structures, examples of which include the Massey products in algebraic topology and Feynman diagrams in mathematical physics. The goal of this talk is to explain the homotopy transfer theorem and touch on some examples.

References: Vallette – Algebra+Homotopy=Operad and references therein

7. L_∞-algebras and mapping spaces in rational homotopy theory

Explain the Deligne-Getzler ∞-groupoid associated to a nilpotent dgla, or more generally, a complete L_∞-algebra and its relation to deformation theory. The main goal of this talk is to discuss Berglund's application to rational models for mapping spaces.

References: Ezra Getzler – Lie theory for nilpotent L_∞-algebras, Berglund – Rational homotopy theory of mapping spaces via Lie theory for L_∞-algebras

Part 3. Examples of formal moduli problems

8. Examples

We'd be happy to help you find examples (and references) you'd like to talk about according to your taste.

Part 4. Koszul duality for associative algebras

9. Koszul duality for associative algebras

Explain the notion of a quadratic algebra and its "dual" with a star role for the example of exterior vs. symmetric algebras. Explain the associated adjunction between module categories (e.g., BGG correspondence). Discuss generalizations (e.g., to quadratic-linear algebras such as universal enveloping algebras) and the relationship with bar-cobar constructions.

References: Priddy – Koszul resolutions; Polishchuk and Positselski – Quadratic Algebras; Loday and Vallette – Chapter 1 of "Algebraic Operads"

10. Koszul duality and equivalences of derived categories

Develop in more depth the relationship between modules over A and modules over $A^!$, particularly the statements at the level of derived and ∞-categories.

Floystad – Koszul duality and equivalences of categories; Keller – Koszul duality and coderived categories (after K. Lefèvre); DAG X, Sections 3.4-5

11. Koszul duality in representation theory

Explain applications of these ideas in the context of representation theory. (There are world experts in Bonn on such things, if you want some better advice than we'll give you.)

References: Beilinson, Ginzburg, and Soergel – Koszul duality patterns in representation theory; Mazorchuk, Ovsienko, and Stroppel - Quadratic duals, Koszul dual functors, and applications

Part 5. Operads

12. Introduction to Operads [Potentially 2 talks]

The goal here is (1) to motivate and state the formal definitions and (2) to describe several important examples. Start concretely and describe operads with values in vector spaces (such as the associative or Lie operads) and topological spaces (such as the little n-cubes operad). Sketch variations on the general definitions (a monadic definition vs symmetric sequences vs ...) and, if time permits, mention some ∞-categorical versions that have recently appeared (but don't worry about doing that part carefully).

References: May – "Operads, algebras, and modules" (`http://www.math.uchicago.edu/~may/PAPERS/mayi.pdf` but see his website for much more) ; Loday and Vallette – Algebraic Operads; Vallette – Algebra + Homotopy = Operad; Lurie – Higher Algebra (the beginning of Chapter 2 has a lovely introduction to ordinary operads and symmetric monoidal categories that leads quite naturally to a quasi-categorical generalization)

13. Koszul duality for operads

Introduce the operadic analog of the Koszul duality of algebras: certain (co)operads are "Koszul dual" and this induces adjunctions between their categories of algebras (which replace the categories of modules over Koszul dual (co)algebras). Explain in some detail the Koszul duality of the commutative and Lie operads and the Koszul self-duality of the associative operad.

References: Loday and Vallette – Algebraic Operads; Ginzburg and Kapranov – Koszul duality for operads; Sinha - Koszul duality in algebraic topology (a survey of a different direction, orienting these ideas vis-à-vis the rest of the course)

14. E_n-operad

Discuss different models of the E_n operads and explain how to compute the associated homology operads in characteristic zero, which are shifted-Poisson operads. Discuss subtleties around other characteristics. Describe the topological motivations for these ideas (i.e., a recognition principle for n-fold based loop spaces) and possibly connections with topological field theory.

References: Sinha – The (non-equivariant) homology of the little disks operad; Cohen, May, Milgram – The Homology of Iterated Loop Spaces; May – Geometry of Iterated Loop Spaces

Other suggestions are welcome!

References

[1] Jacob Lurie. ICM address 2010. available at `www.math.harvard.edu/~lurie/papers/moduli.pdf`.

[2] Jacob Lurie. DAG X: Formal moduli problems. available at `www.math.harvard.edu/~lurie/papers/DAG-X.pdf`.

[3] Jacob Lurie. Higher algebra. available at `www.math.harvard.edu/~lurie/papers/higheralgebra.pdf`.

[4] David Ayala and John Francis. Poincaré/Koszul duality. available at `http://front.math.ucdavis.edu/1409.2478`.