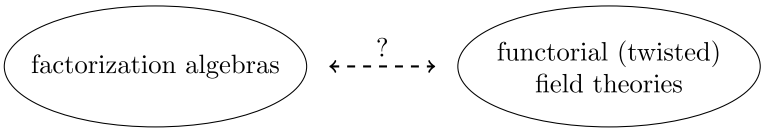
Factorization algebras and (twisted) functorial field theories - the topological case

Claudia Scheimbauer
(joint work with Damien Calaque, Theo Johnson-Freyd)

In recent years, quantum field theories have been studied by mathematicians using, among others, two approaches: *functorial field theories* and their variations following [1, 13, 15] axiomatizing the state space and the partition function; and the more recent approach via *factorization algebras* (cf. [4]) axiomatizing the structure of the observables of perturbative quantum field theories. In this talk, after providing an introduction to both, we will explain how to relate these two concepts in the case of topological field theories, see also [5, 6, 12].



Unravelling the axiomatization by Atiyah, n-dimensional topological field theories (nTFTs) are symmetric monoidal functors out of a suitable category of spacetimes, called *bordisms*, which are n-dimensional topological manifolds, perhaps required to be smooth, or equipped with some tangential structure such as an orientation or a framing. In general, functorial field theories allow for spacetimes endowed with more general types of geometries, such as a conformal or Euclidean structure, encoded as a sheaf $\mathcal{G}$ on the site of n-dimensional manifolds valued in a suitable target category, usually taken to be sets or, to allow for homotopical versions, spaces. Passing to higher categories of cobordisms as defined in joint work with Calaque in [5] leads to (fully) extended field theories which in the topological case describe locality of the field theory: the Cobordism Hypothesis [10] shows that fully extended nTFTs are fully determined by their value at a point.

However, important examples of quantum field theories may not fit into this framework, as can already be seen in Segal's weakly conformal field theories of [13]. This leads to the generalization in form of *twisted field theories* as defined by Stolz-Teichner in [14], which implement the idea that the partition function may not be a number, but rather an element in some line or vector space. A twisted n-dimensional field theory is an (op)lax natural transformation from the trivial field theory (sending everything to the monoidal unit) to an n-dimensional field theory called the *twist*. In the fully extended case, a definition requires a notion of (op)lax natural transformations in the setting of higher categories, as was given in joint work with Johnson-Freyd in [8].

The target of an (extended) nTFT should be taken to be a (delooping of) the category of (dg) vector spaces (n)Vect. An example in the once-extended case is the bicategory of Bimod of algebras, bimodules, and intertwiners. Examples of deloopings of Bimod as higher categories have been constructed in joint work with Calaque [6] and Johnson-Freyd [8].

Factorization algebras should be thought of as multiplicative versions of cosheaves. As such they encode the structure of the observables of perturbative quantum field theories, as we'll see in the lecture series by Ryan Grady, Si Li, and Brian Williams.

Definition. *Let $\mathcal{G}$ be a geometry on n-dimensional manifolds. A $\mathcal{G}$-*factorization algebra *is a symmetric monoidal functor*

$$\mathcal{F} : \text{MFLD}^{\mathcal{G},\amalg} \longrightarrow \mathcal{S}^{\otimes}$$

satisfying descent for Weiss covers. Here, $\text{MFLD}^{\mathcal{G},\amalg}$ *is a category of $\mathcal{G}$-manifolds and $\mathcal{G}$-isometric embeddings.*

If the target $\mathcal{S}$ naturally is a homotopical category, i.e. a symmetric monoidal $(\infty, 1)$-category, this definition should be modified to this setting. When considering a fixed n-dimensional manifold M, a factorization algebra on M does not see any geometry:

Definition. *A* factorization algebra $\mathcal{F}$ on M *is an algebra over the colored operad with open sets in M as colors and*

$$\mathcal{P}reFact_M(U_1, \ldots, U_n; V) = \begin{cases} \{*\} & \text{if } U_1 \amalg \ldots \amalg U_n \subseteq V; \\ \emptyset & \text{otherwise,} \end{cases}$$

satisfying multiplicativity, i.e. $\mathcal{F}(U) \otimes \mathcal{F}(V) \xrightarrow{\simeq} \mathcal{F}(U \amalg V)$*, and descent for Weiss covers.*

We will see several examples and variations appearing throughout the talks this week: conformal nets (Henriques), structures appearing in algebraic quantum field theory (Rejzner), algebro-geometric versions (Cliff), and several topological examples (Kapranov, Knudsen). Topological factorization algebras are obtained from *factorization homology* [9, 2, 3, 7, 11]: the are defined locally and "glued together" using the tangential structure of a manifold. The local data needed for this procedure in the framed case is that of an *E_n-algebra in $\mathcal{S}$*.

Factorization homology is the key ingredient in relating topological factorization algebras and functorial topological field theories. The target of the latter will be a symmetric monoidal Morita-$(\infty, n+1)$-category $\text{ALG}_n^{ptd}(\mathcal{S})$. Its objects are E_n-algebras, morphisms from A to B are E_{n-1}-algebras which are (A, B)-bimodules, 2-morphisms are bimodules of bimodules, ... and n-morphisms are *pointed* bimodules of ... of bimodules. Its $(n+1)$-morphisms are intertwiners, but the non-invertible ones cannot be seen by the n-dimensional theory. This $(\infty, n+1)$-category can be built using factorization algebras on $\mathbb{R}^n$ which satisfy certain constructibility conditions to encode the objects and k-morphisms for $1 \leq k \leq n$. We restrict to explaining the framed case. The main theorem of my thesis [12], of which I will outline the proof in the talk, is the following:

Theorem. *(Calaque-S. [6]) Let $\mathcal{S}$ be a symmetric monoidal $(\infty, 1)$-category which is $\otimes$-sifted-cocomplete. Given any object in $\text{ALG}_n^{ptd}(\mathcal{S})$, i.e. an E_n-algebra A in*

$\mathcal{S}$, the assignment sending a point to A extends to a fully extended framed n-dimensional topological field theory

$$\mathcal{FH}_n(A) : \textsc{Bord}_n^{fr} \longrightarrow \textsc{Alg}_n^{ptd}(\mathcal{S}).$$

Moreover, any fully extended framed nTFT with target $\textsc{Alg}_n^{ptd}(\mathcal{S})$ arises this way.

The target is not a delooping of $\textsc{Vect}$ since its bimodules are pointed. However, forgetting the pointings yields a forgetful functor of $(\infty, n+1)$-categories $\mathcal{U} : \textsc{Alg}_n^{ptd}(\mathcal{S}) \to \textsc{Alg}_n(\mathcal{S})$ and a twisted framed nTFT with twist $T = \mathcal{U} \circ \mathcal{FH}_n(A)$,

$$\textsc{Bord}_n^{fr} \underset{T}{\overset{1}{\Downarrow}} \textsc{Alg}_n(\mathcal{S}).$$

Corollary. *Let $\mathcal{S}$ be a symmetric monoidal $(\infty, 1)$-category which is $\otimes$-sifted-cocomplete. Equivalent data are*

(1) *topological factorization algebras with target $\mathcal{S}$*
(2) *twisted framed nTFTs with target $\textsc{Alg}_n(\mathcal{S})$.*

Remark. *The implication (1) $\Longrightarrow$ (2) in the corollary is the fully extended topological case of a theorem by Dwyer-Stolz-Teichner, which shows that $\mathcal{G}$-factorization algebras valued in the category of chain complexes $\mathcal{S} = \textsc{Ch}$ lead to twisted $\mathcal{G}$-field theories with target the bicategory of dg categories, bimodule categories, and intertwiners. We expect this construction to fully extend using similar methods.*

References

[1] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.
[2] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.
[3] David Ayala, John Francis, and Hiro L. Tanaka. Factorization homology of stratified spaces. arXiv:1409.0848
[4] Kevin Costello and Owen Gwilliam. Factorization algebras in perturbative quantum field theory.
[5] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms. 2015. arXiv:1509.08906, based on the second author's PhD thesis.
[6] Damien Calaque and Claudia Scheimbauer. Factorization homology as a fully extended topological field theory. 2016. In preparation, based on the second author's PhD thesis.
[7] Grégory Ginot, Thomas Tradler, and Mahmoud Zeinalian. Higher Hochschild homology, topological chiral homology and factorization algebras. *Comm. Math. Phys.*, 326(3):635–686, 2014.
[8] T. Johnson-Freyd and C. Scheimbauer. (Op)lax natural transformations, relative field theories, and the "even higher" Morita category of E_d-algebras. 2015. arXiv:1502.06526.
[9] Jacob Lurie. Higher algebra.
[10] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.
[11] Scott Morrison and Kevin Walker. Blob homology. *Geom. Topol.*, 16(3):1481–1607, 2012.
[12] Claudia Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory.* PhD thesis, ETH Zürich, 2014.

[13] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[14] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011.

[15] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.