2-Segal spaces and the Waldhausen construction

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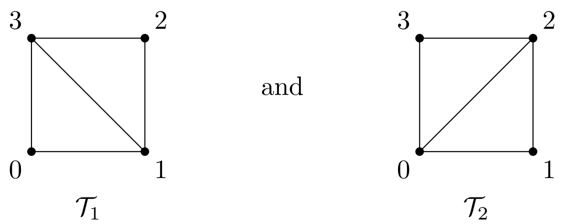
The notion of (unital) 2-Segal objects in a model category or $(\infty,1)$-category was introduced by Dyckerhoff and Kapranov in [1] and, independently, for the $(\infty,1)$-category of spaces under the name of decomposition spaces, by Gálvez-Carrillo, Kock, and Tonks [2]. It is a homotopical variant of a category which has a multi-valued composition: a 2-Segal object is a simplicial object $X_\bullet$ satisfying conditions which are 2-dimensional generalizations of the usual Segal condition and are parametrized by triangulations of regular n-gons for $n \geq 3$. For example, for the two triangulations



of the square the following conditions are required: The triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ determine the two diagrams

$$X_{\{0,1,3\}} \xrightarrow{d_0} X_{\{1,3\}} \xleftarrow{d_1} X_{\{1,2,3\}} \quad \text{and} \quad X_{\{0,1,2\}} \xrightarrow{d_1} X_{\{0,2\}} \xleftarrow{d_2} X_{\{0,2,3\}}$$

which in turn give two maps

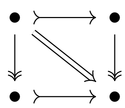
$$f_{\mathcal{T}_1} : X_3 \longrightarrow X_{\{0,1,3\}} \underset{X_{\{1,3\}}}{\times} X_{\{1,2,3\}} \quad \text{and} \quad f_{\mathcal{T}_2} : X_3 \longrightarrow X_{\{0,1,2\}} \underset{X_{\{0,2\}}}{\times} X_{\{0,2,3\}}.$$

The 2-Segal condition requires these maps to be weak equivalences. The 0-simplices X_0 should be thought of as the objects of the multivalued category, X_1 as the morphisms, and the span $X_1 \times_{X_0} X_1 \leftarrow X_2 \to X_1$ as the multivalued composition. An extra condition called unitality ensures that every composition with an identity morphism is (homotopically) unique.

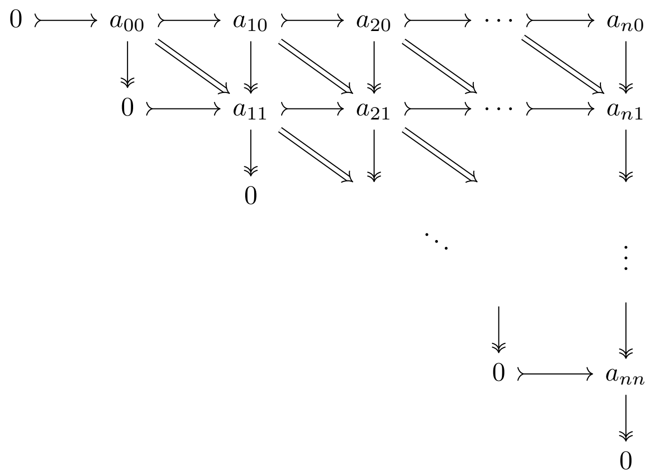
Examples include Segal's nerve of a partial (topological) monoid from [3] and a 2-dimensional cobordism "category" with genus constraints from [6] requiring that the genus of a morphism is $\leq g$ for some fixed $g \in \mathbb{N}$.

Furthermore, both [1] and [2] showed that Waldhausen's $S_\bullet$-construction from [4] provides examples of 2-Segal spaces. In [5] we provide a generalization thereof which proves to be exhaustive in the discrete setting. Let us briefly explain this construction.

The abstract structure needed to define an $S_\bullet$-construction given by diagrams of a certain shape are certain double categories. A double category is a category internal to categories. It has a set of objects, two kinds of morphisms between two objects which we suggestively call "horizontal" and "vertical" morphisms, and 2-morphisms ("squares") which have horizontal source and target morphisms and vertical source and target morphisms:



To define a simplicial object similarly to the one arising from the $S_\bullet$-construction we need the double category to be "pointed", i.e. there is an object 0 which is initial for the horizontal category and terminal for the vertical category. For such a pointed double category $\mathcal{D}$ we let $S_k(\mathcal{D})$ be the set of diagrams of the form



This gives a simplicial set whose face maps are given by deleting a row and column. This simplicial set is 2-Segal if we started with a double category which is "stable", meaning that any 2-morphism is uniquely determined by its horizontal and vertical sources, and also is uniquely determined by its horizontal and vertical targets. Note that $S_0(\mathcal{D}) = \{0\}$. We call a 2-Segal set with this property "reduced".

More generally, we can replace the condition that the double category should be pointed by the data of an "augmentation", which is a certain subset of objects. Then we require the elements on the diagonal to be in the augmentation.

Finally, this generalized $S_\bullet$-construction leads to an equivalence, whose inverse essentially is given by the décalages of the simplicial set.

Theorem (BOORS). (1) *The generalized $S_\bullet$-construction is an equivalence of categories between pointed, stable double categories and reduced unital 2-Segal sets.*

(2) *The generalized $S_\bullet$-construction is an equivalence of categories between augmented, stable double categories and unital 2-Segal sets.*

A generalization of the above theorem to more homotopical settings such as 2-Segal spaces is work in progress.

References

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[5] J. E. Bergner, A. Osorno, V. Ozornova, M. Rovelli, and C. Scheimbauer. 2-Segal sets and the Waldhausen construction. In preparation.

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