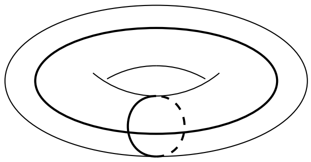
Miniprojects

1. Let $T^2 = S^1 \times S^1$ be the torus. We show that every continuous map $S^2 \to T^2$ is nullhomotopic, but that there are non-nullhomotopic maps $T^2 \to S^2$.

   (a) Let $\pi\colon \mathbb{R} \to S^1$ be the universal cover of the circle. Show that $\pi \times \pi\colon \mathbb{R}^2 \to S^1 \times S^1 = T^2$ is the universal cover of the torus.

   Show that every continuous map $f\colon S^2 \to T^2$ lifts to the universal cover. Deduce that f is nullhomotopic.

   (b) Let $X = (S^1 \times \{p\}) \cup (\{p\} \times S^1) \subset T^2$ for some point $p \in S^1$. X is represented by a thick line in the following picture.

   Show that T^2/X is homeomorphic to S^2.

   Show that the projection map $\pi\colon T^2 \to T^2/X \cong S^2$ has degree[1] 1, and deduce that it is not nullhomotopic.

   [Hint: Decompose the torus as a union $T^2 = A \cup B$ where A is an open disc disjoint from X, and B is a small open neighbourhood of X. Then $T^2/X = S^2$ decomposes as a union of the open discs $\pi(A)$ and $\pi(B)$. Compute the degree of π from the Mayer–Vietoris sequences of $T^2 = A \cup B$ and $S^2 = \pi(A) \cup \pi(B)$.]

2. Consider the open cover of the double torus, $\Sigma_2 = U \cup V$, where U and V are both $T^2 \setminus D$ (D is a disk). Use Mayer-Vietoris to compute the homology of Σ_2. Compare your result to the result we saw in class where we used Van Kampen's Theorem.

3. Choose any of the exercises you have not done, and work it out!

[1]We have seen in class that $H_2(T^2) = \mathbb{Z} = H_2(S^2)$. We say that a map $f\colon T^2 \to S^2$ has degree d if the induced map $H_2(f)\colon H_2(T^2) \to H_2(S^2)$ takes $1 \in H_2(T^2)$ to $d \in H_2(S^2)$.