EAUMP-ICTP Summer School on Topology in Uganda 2019

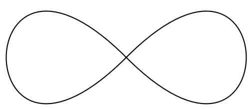
Manifolds Claudia Scheimbauer

Exercises

Manifolds via examples.

1. *(stereographic projection)* Show that the stereographic projection
$$\mathbb{R}^2 \supset S^1\backslash\{(0,1)\} \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{x}{1-y}$$
is a homeomorphism. Find the stereographic projection for $S^1\backslash\{(0,-1)\}$ and conclude that S^1 is a manifold.

2. Consider the figure eight



Explain why this is not a manifold. *Hint: Look at a neighborhood of the crossing point and remove the crossing point.*

3. *(quotient topology)* Let $\pi : X \to X/\sim$ be a quotient map. Define a subset $U \subset X/\sim$ to be open if $\pi^{-1}(U)$ is open in X. Show that this defines a topology on $X/\sim$.

4. Consider the line with a double point at zero; that is, take $\big(\mathbb{R}\times\{0,1\}\big)/\sim$, where for any $x \in \mathbb{R}\backslash\{0\}$, we have $(x,0)\sim(x,1)$. Show that this satisfies all conditions for a manifold except for Hausdorff.

5. Show that the quotient of the square $[0,1]\times(0,1)$ by the equivalence relation $(0,x)\sim(1,1-x)$ is a manifold. Explain why the quotient is homeomorphic to the Moebius band as defined in class, namely, as
$$M/x \sim -x\,,$$
where $M \subset S^2$ is a "band" around the equator of S^2.

6. Let M be a manifold and G a discrete group acting on M. Show that the following are equivalent.
   (a) The multiplication $G\times M \to M,\quad (g,x)\mapsto g\cdot x$ is a continuous map of topological spaces
   (b) for every $g\in G$, the map $l_g\colon M\to M,\quad x\mapsto g\cdot x$ is a continuous map of topological spaces.

7. Let $G=\mathbb{Z}/2\mathbb{Z}\cong\{1,-1\}$ act on $M=(-1,1)$ by multiplication. Show that M/G is not a manifold.

8. Let $G = \mathbb{R}$ act on $M = S^1 \times \mathbb{R}$ as follows:

$$(\alpha, (e^{2\pi i x}, y)) \longmapsto (e^{2\pi i(x+\alpha y)}, y).$$

Show that M/G is not a manifold.

9. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$.

10. Let M be a manifold and G a finite group acting on M. Show that if the action is free, then for every $x \in M$ there is an open neighborhood U such that for every $g \in G \backslash \{e\}$ we have that $U \cap gU = \varnothing$.

11. Show that in the situation of the Quotient Manifold Theorem, the projection $\pi : M \to M/G$ is a covering map.

12. Let $f : E \to B$ be a covering with countable fibers and B a manifold. Show that E is a manifold.

Surfaces from polygons.

1. Convince yourself that one indeed gets a manifold from a gluing pattern.

2. Glue a disk onto a Moebius band by identifying their unique edges. Show that this gives projective 2-space $\mathbb{RP}^2$.

3. Show that the pasting scheme $abcb^{-1}a^{-1}c^{-1}$ gives a torus.

4. Show that the pasting scheme from the video $abcda^{-1}b^{-1}c^{-1}d^{-1}$ gives Σ_2.

5. *Klein bottle*

(a) Repeat the argument that the pasting pattern $aadd$ also defines the Klein bottle (this was called 2-fold projective plane P_2 and was done in class).

(b) Moreover, show that one can get the Klein bottle as a quotient of $M = S^1 \times S^1$ by the group $\mathbb{Z}/2\mathbb{Z}$ acting by $(z, w) \mapsto (1/z, -w)$. Here we consider $z, w \in S^1 \subset \mathbb{C} \cong \mathbb{R}^2$.

6. For the genus g surface Σ_g, choose a triangulation and compute the Euler characteristic as in Ulrike's class. Compare with others: if they chose a different triangulation, did they get a different number?

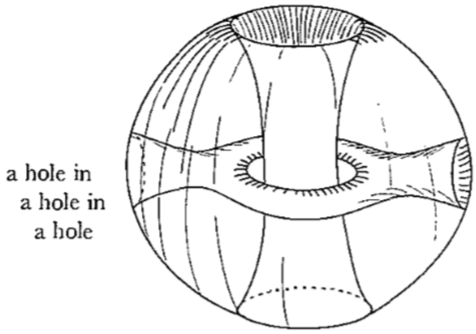
More difficult extra question: how can you find a triangulation of the surface starting from a pasting scheme? *Hint: Subdivide!* Use this to compute the Euler characteristic of P_k as well.

7. Compute the fundamental group of $\mathbb{RP}^2$ using Van Kampen's Theorem (and possibly a pasting scheme).

8. Compute the fundamental group of the k-fold projection plane P_k using Van Kampen's Theorem. As a corollary, show that its first homology is $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$. Conclude that all P_k and Σ_g are pair-wise non-homeomorphic.

Classification of surfaces and other facts.

1. *A hole in a hole in a hole* What is the genus of a *hole in a hole in a hole*, depicted in the figure below, taken from Michael Spivak's book *A Comprehensive Introduction to Differential Geometry*:



2. *Three Utilitities Problem* Assume we have three houses and would like to connect them to gas, water, and power. However, we do not want any lines to cross each other. Is this possible if we are living in a plane? What if we live on a sphere or a torus?

3. Use the Borsuk-Ulam Theorem[1] to prove the following statement:

   Proposition. *For any cover A_1, A_2, A_3 of S^2 by 3 closed sets, there is at least one set which contains two antipodal points.*

   Hint: Use the distance function $\mathrm{dist}(x, A_i) = \min_{a \in A_i} |x - a|$.

4. Use the Hairy Ball Theorem[2] to prove the following statement:

   Proposition. *Any continuous function that maps S^2 into itself has either a fixed point or a point that maps onto its own antipodal point.*

   Hint: Use the stereographic projection to produce a vector field.

[1] **Borsuk-Ulam Theorem.** For every continuous map $f : S^2 \to \mathbb{R}^2$ there is a pair of antipodal points x and $-x$ such that $f(x) = f(-x)$.

[2] **Hairy Ball Theorem.** There is no non-vanishing vector field on S^2.

5. We call a compact connected one-dimensional manifold a *curve*. Prove a classification of curves following steps similar to the classification of surfaces:

   (a) *(combinatorial structure: triangulation)* Prove that any curve is homeomorphic to a graph as follows: use compactness to cover with a finite set of closed intervals and proceed by induction on the number of pairs of intervals that overlap in their interior.

   (b) *(simplifying the combinatorial structure by an inductive procedure)* Use the first part to show that vertices in the graph structure connect exactly two edges by removing a vertex in the graph. By induction, show that the curve is homeomorphic to a polygon. Conclude that any curve is a circle.

Smooth manifolds.

1. Go back and check whether the examples of manifolds we have seen so far were smooth or not.

2. Compute the degrees of the following maps:

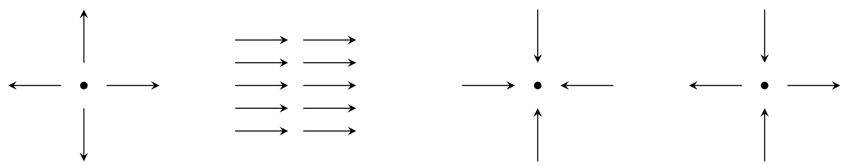
   (a) $f : S^1 \to S^1, z \mapsto -z$,

   (b) $f : S^1 \to S^1, z \mapsto \bar{z}$,

   (c) $f : S^1 \to S^1, z \mapsto z^n$,

   (d) $f : S^n \to S^n, z \mapsto -z$.

3. Compute the index at the singularity of the following vector fields



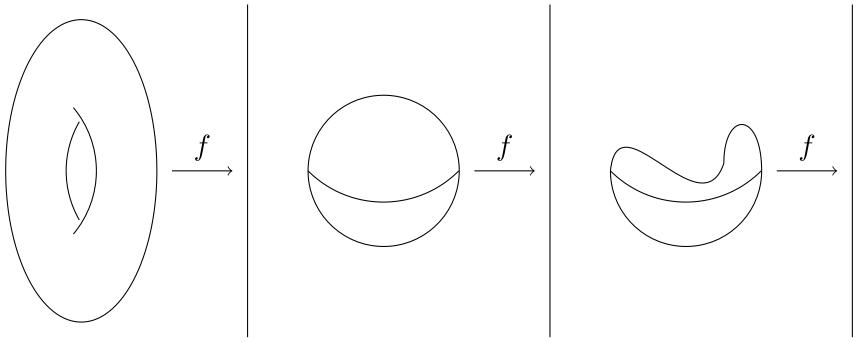
4. Find a non-vanishing vector field on $S^1 \times S^1$. Use the Poincaré-Hopf Theorem[3] to compute the Euler characteristic of the torus. Addendum: can you find, for every $(k, l) \in \mathbb{Z}^2$, a non-vanishing vector field which are all pair-wise different?

5. Compute the Morse complex for $f : S^1 \to \mathbb{R}, (x, y) \mapsto y$ and compare to the homology computation from Ulrike's class.

6. Let $f : M \to \mathbb{R}$ be a Morse function and let $c_i(f)$ be the number of critical points of f with index i. Verify the following theorem in the examples below.

   Theorem. (Morse's Theorem) *Let M be smooth and compact and $f : M \to \mathbb{R}$ a Morse function. Then*
   $$\chi(M) = \sum_i (-1)^i c_i(f).$$

[3]**Poincaré-Hopf Theorem.** Let v be a vector field on a compact smooth manifold with a finite set of isolated singularities. Then $\sum_i \text{index}_{x_i}(v) = \chi(M)$.



7. Look up something from class that sparked your interest and would like to learn more about. For example, start with wikipedia and go to the references therein. Or check out some of the links on the website `folk.ntnu.no/claudiis/TopologyUganda`

Thank you for attending! **Weebale Nnyo!**