A factorization view on states and observables

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States versus observables

States
(Schrödinger picture)

Observables
(Heisenberg picture)

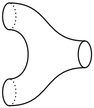
States versus observables

States

Functorial field theory
(from TFT/CFT)

Atiyah, Segal, Freed, Lurie, ...

$$\mathrm{Bord}^{\amalg} \to \mathrm{Vect}^{\otimes}$$



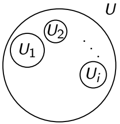
$$\mapsto (V \otimes V \to V)$$

Observables

Factorization algebras

Beilinson-Drinfeld, Lurie, Costello-Gwilliam,

Morrison-Walker, Ayala-Francis, ...



$$\rightsquigarrow \bigotimes_{i \in I} \mathcal{F}(U_i) \to \mathcal{F}(U)$$

States – functorial topological field theories

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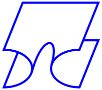
$$\text{Bord} \longrightarrow \text{Vect}$$

 $\longmapsto$ linear map

$\bullet \quad \bullet \quad \longmapsto$ vector space

States – functorial topological field theories

Bord	$\longrightarrow$	Vect	$2\mathrm{Vect} = \mathrm{Alg}_1$
	$\longmapsto$		homomorphism
	$\longmapsto$	linear map	bimodule
• •	$\longmapsto$	vector space	algebra

States – functorial topological field theories

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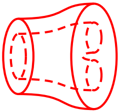
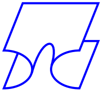
Bord	$\longrightarrow$	Vect	$2\mathrm{Vect} = \mathrm{Alg}_1$	$3\mathrm{Vect} = \mathrm{Alg}_2$
	$\longmapsto$			homomorphism
	$\longmapsto$		homomorphism	bimodule of bimodules
	$\longmapsto$	linear map	bimodule	bimodule
• •	$\longmapsto$	vector space	algebra	E_2-algebra

The target (higher) category

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Construction (Calaque–S., Haugseng, Johnson-Freyd–S.)

Given a "nice" symmetric monoidal (∞, k)*-category* $\mathcal{S}$*, there is a symmetric monoidal* $(\infty, n+k)$*-category* $\mathrm{Alg}_n(\mathcal{S})$*.*

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Application

$\mathcal{S} = \mathrm{Cat}_k$ = k-linear categories, k-linear functors, natural transformations (is also a 2Vect):

- $\mathrm{Alg}_1(\mathrm{Cat}_k)$ is natural home for tensor categories (cf. Turaev-Viro theory)
- $\mathrm{Alg}_2(\mathrm{Cat}_k)$: objects are braided monoidal categories, e.g. $\mathrm{Rep}_q(\mathfrak{g})$.

Cobordism Hypothesis and finiteness conditions

Hopkins-Lurie, Lurie, Ayala-Francis

$$
\begin{array}{ccc}
\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, n\mathrm{Vect}) & \xrightarrow{\;ev_{pt}\;} & n\mathrm{Vect} \\
 & & \uparrow \\
 & gpd(n\mathrm{Vect}^{n\text{-d}}) \hookrightarrow &
\end{array}
$$

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"*n*-dualizable" (over $\mathbb{C}$):

- $n = 1$: finite dimensional vector space
- $n = 2$: finite dimensional semi-simple algebra Lurie, Pstragowski
- $n = 3$, $\mathrm{Alg}_1(\mathrm{Cat}_k)$: finite semi-simple tensor category; in particular, fusion category Douglas–Schommer-Pries–Snyder

A relative version: twisted field theories

(not to be confused with boundary "relative" field theories)

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Stolz-Teichner:

$$\mathrm{Bord}_n \;\underset{T}{\overset{S}{\Downarrow Z}}\; (n+1)\mathrm{Vect}$$

with either S or T the trivial theory $\mathbb{1} = k$, the other is the "twist".
On closed manifold M: get $k \to T(M)$ (a *vector* in the vector space $T(M)$) or $S(M) \to k$ (a *covector* in the vector space $S(M)$).

Technically: *lax* or *oplax* natural transformation Johnson-Freyd–S.

Proposition (Johnson-Freyd–S.)

1-dimensional twisted topological field theories with target Alg_1 *are fully determined by a morphism* ${}_AM_B$ *which has*

(lax) *a left adjoint, i.e. is finitely presented and projective over* A*, or*

(oplax) *a right adjoint, i.e. is finitely presented and projective over* B*.*

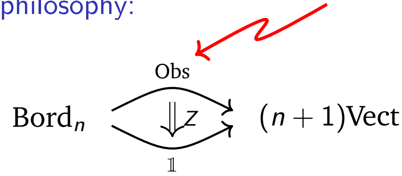
Example (Gwilliam–S.)

Take a (possibly infinite dimensional) vector space V, and view it as a bimodule ${}_{\mathrm{End}\,V}V_k$. This *always* determines a lax twisted theory, and an oplax twisted theory iff V is finite dimensional.

Observables versus states revisited

factorization algebra of observables/point operators

Stolz-Teichner's philosophy:



Can think of Z as the "trace".

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factorization algebra of observables/point operators

Stolz-Teichner's philosophy:

$$\mathrm{Bord}_n \underset{\mathbb{1}}{\overset{\mathrm{Obs}}{\rightrightarrows}} (n+1)\mathrm{Vect}, \qquad \Downarrow Z$$

Can think of Z as the "trace".

Topological case: The twist arises from factorization homology of an E_n-algebra, with target $(n+1)\mathrm{Vect} = \mathrm{Alg}_n$ Calaque-S.

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Theorem (S.)

The factorization model of the $(\infty, 2)$*-category* Alg_2 *is fully 2-dualizable.* *(="has duals" Lurie = "has adjoints" Francis)*

proof

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Theorem (S.)

The factorization model of the $(\infty, 2)$-category Alg_2 is fully 2-dualizable. *(="has duals" Lurie = "has adjoints" Francis)*

proof

Proposition (Gwilliam–S. after Johnson-Freyd–S.)

2-dimensional twisted topological field theories with target Alg_2 are fully determined by a morphism ${}_SM_T$ for which the unit and counit of the adjunction between M and its left adjoint have left adjoints.

(This holds iff the same statement with "right" holds.)

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Deligne's Conjecture:

Given an algebra A, its Hochschild cohomology $Z(A)$ is an E_2-algebra. Moreover, it acts on A.

$\Rightarrow$ bimodule ${}_{Z(A)}A_k$: generalization of ${}_{\mathrm{End}\,V}V_k$ from above, since $Z(A)$=derived endomorphisms of A as an (A, A)-bimodule =derived center of A.

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Theorem (Gwilliam-S.)

${}_{Z(A)}A_k$ determines a twisted field theory iff A is smooth and proper over $Z(A)$. Explicitly, this means that

- *A has a left adjoint as a $(Z(A), m_1^{op})$-algebra*
- *A has a left adjoint as a "$A \otimes_{Z(A)} A^{op}$-algebra".*

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Example

Underived situation: A=polynomial differential operators (Weyl algebra) in characteristic p, $Z(A)$=usual center of A. Then,

- A is finitely presented and projective over $Z(A)$
- A is separable over $Z(A)$.

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cf B-model M variety, $Coh(M)$ dg category of coherent sheaves is 2-dualizable if M is smooth and proper.
Modifications of above would just need: smooth and proper over $HH^*(Coh(M)) = \Gamma(\Lambda T_M)$ (polyvector fields) (as factorization algebra: Li-Li)

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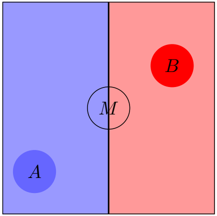
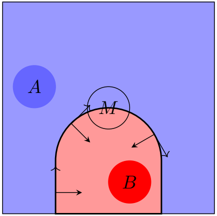
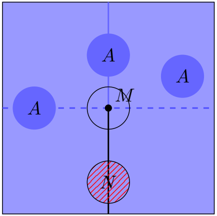
“Proof” of Theorem of existence of adjoints for 1-morphisms:



1-morphism



bend right



counit of left adjoint