The higher category of E_n-algebras and factorization homology as a fully extended topological field theory

Claudia Scheimbauer

(Homotopy) algebras and bimodules over them can be viewed as factorization algebras on $\mathbb{R}$ which are locally constant (with respect to a certain stratification). Moreover, Lurie proved that E_n-algebras are equivalent to locally constant factorization algebras on $\mathbb{R}^n$. Starting from this I will explain how to model the Morita category of E_n-algebras as an (∞, n)-category. Every object in this category, i.e. any E_n-algebra A, is "fully dualizable" and thus gives rise to a (fully extended, in the sense of Lurie) TFT by the cobordism hypothesis of Baez-Dolan-Lurie. It turns out that this TFT can be explicitly constructed by (essentially) taking factorization homology with coefficients in the E_n-algebra A.

1 (Homotopy) algebras and bimodules as factorization algebras

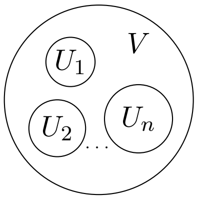
1.1 What is a factorization algebra?

Factorization algebras were, inspired by the algebraic counterpart from Beilinson-Drinfeld, introduced by Costello-Gwilliam in [CG14] as the structure of observables of a quantum field theory. They can be thought of as "multiplicative cosheaves" on a topological space.

Definition 1.1. Let X be a topological space and let $\mathcal{C}$ be a symmetric monoidal category with weak equivalences. A *prefactorization algebra* $\mathcal{F}$ on X (with values in $\mathcal{C}$) is a functor

$$\mathcal{F} : \mathrm{Open}(X) \longrightarrow \mathcal{C}$$

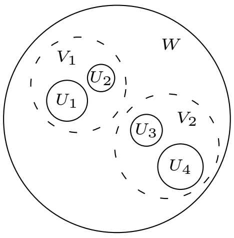
such that for every $U_1 \amalg \ldots \amalg U_n \subseteq V$, we have a morphism



$\rightsquigarrow$

$$f_{U_1,\dots,U_n;V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V).$$

These morphisms have to satisfy the following coherence condition: if $U_1 \amalg \cdots \amalg U_{n_i} \subseteq V_i$ and $V_1 \amalg \cdots \amalg V_k \subseteq W$, the following diagram commutes.



(for $k = n_1 = n_2 = 2$)

$\rightsquigarrow$

$$\begin{array}{ccc} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) & \longrightarrow & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

A *factorization algebra* on M is a prefactorization algebra on M which additionally satisfies a gluing condition saying that given an open cover $\{U_i\}$ of V satisfying certain conditions, $\mathcal{F}(V)$ can be recovered from the $\mathcal{F}(U_i)$'s.This glueing condition is analogous to the one for (homotopy) (co-)sheaves.

For the exact gluing condition and more details on the theory of factorization algebras we refer to loc.cit. and to [Gin].

In particular, for $U \amalg V \subseteq W$, there is a weak equivalence

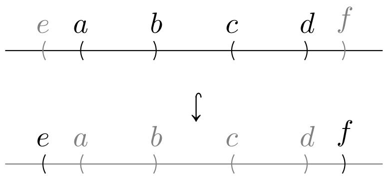
$$\mathcal{F}(U \amalg V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V). \tag{1}$$

1.2 Algebras as factorization algebras

Let A be an associative algebra over a field $\mathbb{K}$ (of char $\mathbb{K} = 0$) and let $X = \mathbb{R}$. Then the following assignment extends by (1) to a factorization algebra $\mathcal{F}_A$:

$$(a, b) \longmapsto \mathcal{F}_A((a, b)) := A,$$

The morphisms $f_{U_1,\dots,U_n;V}$ are defined by multiplication in A, i.e. if $(a, b) \amalg (c, d) \subseteq (e, f)$ with $e < a < b < c < d < f$, the morphism $f_{(a,b),(c,d);(e,f)}$ is given by



$\rightsquigarrow$

$$\begin{array}{ccccc} \mathcal{F}_A((a,b)) & \otimes & \mathcal{F}_A((c,d)) & \longrightarrow & \mathcal{F}_A((e,f)) \\ \| & & \| & & \| \\ A & \otimes & A & \xrightarrow{m} & A \end{array}$$

Associativity ensures the coherence condition.

1.3 Locally constant factorization algebras

Definition 1.2. A factorization algebra is *locally constant* if for any $U \subset V$ which are weakly equivalent, $\mathcal{F}(U) \simeq \mathcal{F}(V)$.

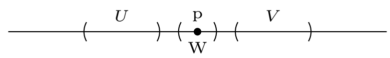
Given a locally constant factorization algebra $\mathcal{F}$ on $\mathbb{R}$, let $A := \mathcal{F}(\mathbb{R})$. Then A is an algebra (up to homotopy), where multiplication is defined by the morphisms f

$$\begin{array}{ccccc} \mathcal{F}((a,b)) & \otimes & \mathcal{F}((c,d)) & \xrightarrow{f} & \mathcal{F}((e,f)) \\ \wr\| & & \wr\| & & \wr\| \\ A & \otimes & A & \longrightarrow & A \end{array}$$

i.e. A is an E_1-algebra.

1.4 Bimodules as factorization algebras

Let A, B be associative algebras over $\mathbb{K}$, M an (A, B)-bimodule, and let $m \in M$ be a distinguished point. Then the following extends to a factorization algebra $\mathcal{F}_M$ on $\mathbb{R}$: Fix a point $p \in \mathbb{R}$. For



$$U \longmapsto \mathcal{F}_M(U) = A, \qquad V \longmapsto \mathcal{F}_M(V) = B,$$
$$p \in W \longmapsto \mathcal{F}_M(W) = M.$$

The factorization maps are given by the bimodule structure and by

$$A \otimes B \longrightarrow M,$$

$$(a, b) \longmapsto amb.$$

This special case comes from the fact that factorization algebras define pointed objects, as we can always include the empty set into any other open set, thus the inclusion $\emptyset \subseteq W$ induces a map

$$\mathbb{K} \longrightarrow M,$$

$$1 \longmapsto m.$$

1.5 Factorization algebras which are locally constant with respect to a stratification

For the full definition of a factorization algebra which is locally constant with respect to a stratification see [Gin]. For this talk, it is enough to consider the following special case:

Definition 1.3. A factorization algebra on $\mathbb{R}$ which is locally constant with respect to a stratification of the form $\mathbb{R} = \{\mathrm{pt}\} \amalg (\mathbb{R}\backslash \mathrm{pt})$ is a factorization algebra $\mathcal{F}$ on $\mathbb{R}$ such that

- $\mathcal{F}|_{\mathbb{R}\backslash\{\mathrm{pt}\}}$ is locally constant,
- if $\{\mathrm{pt}\} \in U \subseteq V$ and $U \simeq V$, then $\mathcal{F}(U) \simeq \mathcal{F}(V)$.

The factorization algebra $\mathcal{F}_M$ defined by a bimodule M as above is locally constant with respect to the stratification $\mathbb{R} = \{p\} \amalg (\mathbb{R}\backslash\{p\})$.

Conversely, any factorization algebra $\mathcal{F}$ which is locally constant with respect to a stratification of the above form determines a homotopy bimodule M over homotopy algebras A, B by setting

$$M := \mathcal{F}(\mathbb{R}),$$
$$A := \mathcal{F}((-\infty, \mathrm{pt})), \qquad B := \mathcal{F}((\mathrm{pt}, \infty)).$$

2 The $(\infty, 1)$-category of E_1-algebras

As a model for $(\infty, 1)$-algebras we use complete Segal spaces, see e.g. [Rez01].

First note that factorization algebras form a category. Taking the nerve of this category we get a simplicial set of factorization algebras. Now the complete Segal space FAlg_1 of E_1-algebras consists of the following data:

- A simplicial set of objects: We take the simplicial subset of the simplicial set of factorization algebras whose vertices are
$(\mathrm{FAlg}_1)_0 = \{E_1\text{−algebras}\} = \{\text{locally constant factorization algebras on } \mathbb{R}\}$
- A simplicial set of morphisms: We take the simplicial subset of the simplicial set of factorization algebras whose vertices are

$$(\mathrm{FAlg}_1)_1 = \left\{ \begin{array}{l} \text{factorization algebras on } \mathbb{R} \text{ which are locally constant} \\ \text{wrt a stratification of the form } \{\mathrm{pt}\} \amalg (\mathbb{R}\backslash \mathrm{pt}) \end{array} \right\}$$

  We get two maps $s, t : (\mathrm{FAlg}_1)_1 \rightrightarrows (\mathrm{FAlg}_1)_0$ sending the factorization algebra to its restriction to $(-\infty, \mathrm{pt}) \simeq \mathbb{R}$, $(\mathrm{pt}, \infty) \simeq \mathbb{R}$, respectively. By the previous section, morphisms correspond to (homotopy) bimodules M over (homotopy) algebras A, B. The maps s, send the (A, B)-bimodule M to A and B, respectively.

- Simplicial sets $(\mathrm{FAlg}_1)_n$: Simplicial subset of the simplicial set of factorization algebras whose vertices are

$$(\mathrm{FAlg}_1)_n = \left\{ \begin{array}{l} \text{factorization algebras on } \mathbb{R} \text{ which are locally constant} \\ \text{wrt a stratification of the form } \{p_1, \ldots, p_n\} \amalg (\mathbb{R}\backslash\{p_1, \ldots, p_n\}) \end{array} \right\}$$

This should be thought of the (classifying) space of n composable morphisms. Composition of morphisms in general is not unique anymore, but the space of these choices must be contractible.

Similarly to above, one can see that such a factorization algebra corresponds to the data of (homotopy) algebras $A_0, \ldots, A_n$ and (A_{i-1}, A_i)-(homotopy)-bimodules M_i.

Moreover, we have $(n+1)$ maps $(\mathrm{FAlg}_1)_n \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} (\mathrm{FAlg}_1)_{n-1}$ (making FAlg_n into a bisimplicial set) defined by either forgetting A_0, M_1 or M_n, A_n, or by "collapsing" two consecutive bimodules to $M_i \otimes_{A_i} M_{i+1}$, which is an (A_{i-1}, A_{i+1})-module.

3 The (∞, n)-category of E_n-algebras FAlg_n.

The following theorem is due to Lurie:

Theorem 3.1. *There is an equivalence of* $(\infty, 1)$*-categories between*

$$\{\textit{locally constant factorization algebras on } \mathbb{R}^n\} \quad \longleftrightarrow \quad \{E_n\textit{-algebras}\}.$$

Remark 3.2. If you don't know what an E_n-algebra is, take this as a definition for this talk.

We will use this theorem to model the higher category of E_n-algebras as an n-fold complete Segal space FAlg_n using factorization algebras which are locally constant with respect to a certain stratification.

For brevity, I will only explain the simplicial sets of k-morphisms:

- objects are E_n-algebras, i.e. locally constant factorization algebras on $\mathbb{R}^n$.
- 1-morphisms are factorization algebras which are locally constant with respect to the stratification consisting of a hyperplane in $\mathbb{R}^n$ orthogonal to the first coordinate axis.
- 2-morphisms are factorization algebras which are locally constant with respect to the stratification consisting of a hyperplane H in $\mathbb{R}^n$ orthogonal to the first coordinate axis and an $(n-2)$-dimensional affine subspace lying in H orthogonal to the second coordinate axis.

etc.

4 Factorization homology as a fully extended TFT

Definition 4.1. A *fully extended topological field theory (TFT)* is a symmetric monoidal functor

$$\mathrm{Bord}_n \longrightarrow \mathcal{C},$$

where $\mathcal{C}$ is a symmetric monoidal (∞, n)-category and Bord_n is "the" (∞, n)-category of bordisms.

In his expository paper [Lur09], Lurie gave a sketch of proof of the following
Theorem 4.2. *(Cobordism Hypothesis of Baez-Dolan) A fully extended TFT is completely determined by its value at a point, which is a "fully dualizable" object in its target category $\mathcal{C}$.*

Main theorem

We would like to construct a fully extended topological field theory whose value at a point is a given E_n-algebra.
Problem 1. Make sense of the symmetric monoidal (∞, n)-category of bordisms. This is done in [CSb].
Problem 2. Find suitable target symmetric monoidal (∞, n)-category $\mathcal{C} = \mathrm{FAlg}_n$, which we explained in the previous sections.

The main result of my thesis is the following theorem, which was first formulated by Lurie in [Lur09].
Theorem 4.3. *[CSa] Any object in FAlg_n, i.e. any E_n-algebra A is fully dualizable. The fully extended TFT determined by A explicitely is given by (essentially) taking factorization homology with coefficients in A,*

$$\begin{array}{ccc} \mathrm{Bord}_n & \longrightarrow & \mathrm{FAlg}_n, \\ (M \xrightarrow{\pi} \mathbb{R}^n) & \longmapsto & \pi_*\left(\int_M A\right). \end{array}$$

References

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