OPERADS, FACTORIZATION ALGEBRAS, AND (TOPOLOGICAL) QUANTUM FIELD THEORY WITH A FLAVOR OF HIGHER CATEGORIES

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Abstract. The main goal of this Lecture Course is to give an introduction to certain algebraic structures arising in the study of (topological) quantum field theory.

Operads, introduced by May, form a framework to abstract families of composable functions in a systematic way and have become appeared in many fields of mathematics, such as topology, homological algebra, algebraic geometry, and mathematical physics. We will encounter algebraic examples such as the associative and commutative operad (encoding associative and commutative algebras) and examples of a topological nature, called the little n-disks operad (encoding the structure of n-fold loop spaces).

Factorization algebras, first introduced by Beilinson and Drinfeld in an algebro-geometric context, are algebraic structures which give a way to encode the structure of observables of a perturbative quantum field theory. Examples include (homotopy) algebras and (pointed) bimodules, but also braided monoidal categories such as the category of finite dimensional representations of a reductive algebraic group $\mathrm{Rep}\, G$ or of the associated quantum group $\mathrm{Rep}\, U_q(g)$. In the situation when the field theory is topological, we essentially get E_n-algebras, which are algebras for the little n-disks operad mentioned above.

A connection of the above concepts to topological field theories is given by factorization homology. We will see how higher category techniques enter the picture and, time permitting, see how this leads to an example of a (fully extended) topological field theory à la Atiyah-Segal. Possible topics for an outlook to ongoing research are so-called twisted field theories in the sense of Stolz-Teichner, which are related to boundary field theories, or factorization homology for higher categories after Ayala-Francis-Rozenblyum.

In these lectures we will introduce some algebraic and categorical structures appearing in mathematical physics, more precisely, the mathematical study of (topological) quantum field theories. We will first recall Atiyah's axiomatic approach to topological field theories using categorical language. We will encounter operads as a framework useful to abstracting algebraic structures. We will in particular be interested in the little n-disks operad, whose algebras are a local model for the observables of a perturbative topological field theory. This connects to the notion of a factorization algebra, which is an abstraction of the algebraic structure of the observables of a, not necessarily topological, perturbative quantum field theory. Factorization algebras lead to (extended) Atiyah-Segal type field theories, we will see a topological example of such given by factorization homology.

1. Recollection: Topological quantum field theories and a first higher flavor

We first recall Atiyah-Segal's axiomatization of topological quantum field theories after [1, 2].

Definition 1.1. *Let $n\mathrm{Cob}$ be the following symmetric monoidal category. Its objects are $(n-1)$-dimensional closed manifolds. A* cobordism *from an object M to another object N is an n-dimensional manifold Σ with boundary together with a diffeomorphism $\partial\Sigma \cong M \amalg N$. A morphism in $n\mathrm{Cob}(M,N)$ is a diffeomorphism class of cobordisms from M to N. Composition of morphisms is given by gluing cobordisms along the common boundary – note that this is well-defined up to the choice of a diffeomorphism. The symmetric monoidal structure is given by disjoint union of manifolds.*

Remark 1.2. There are several variations on this definition: one can take either topological, smooth, or piece-wise linear manifolds; one can also consider manifolds with extra structure such as a tangential structure like an orientation or a framing, or equipments such as bundles on them. Relaxing the condition on being topological, we may consider more general geometric structures like a Euclidean or conformal structure, as in Segal's definition of conformal field theories. In all cases, diffeomorphisms are required to preserve the structure.

Definition 1.3. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal category. An* n-dimensional topological field theory (nTFT) *with values in $\mathcal{C}^\otimes$ is a symmetric monoidal functor*

$$n\mathrm{Cob}^{\amalg} \longrightarrow \mathcal{C}^\otimes.$$

Example 1.4. In Atiyah's original definition, $\mathcal{C}^{\otimes}$ was taken to be Mod_{Λ}, the category of modules for some ground ring Λ with their tensor product. In particular, often one is interested in the case of Λ being a field $\mathbb{K}$, and thus $\mathcal{C}^{\otimes} = \mathrm{Vect}_{\mathbb{K}}$. Variations include graded vector spaces and chain complexes.

In dimensions one and two, a full classification is obtained from the classification of 1- and 2-dimensional manifolds with boundary:

Theorem 1.5. (1) *There is an equivalence between (the category of) oriented 1TFTs and (the groupoid of) dualizable objects in* $\mathcal{C}^{\otimes}$ *given by evaluation at a point. For* $\mathcal{C}^{\otimes} = \mathrm{Vect}_{\mathbb{K}}$*, being dualizable unravels to being finite dimensional.*

(2) *There is an equivalence between (the category of) oriented 2TFTs and (the groupoid of) commutative Frobenius objects in* $\mathcal{C}^{\otimes}$ *given by evaluation at a circle. For* $\mathcal{C}^{\otimes} = \mathrm{Vect}_{\mathbb{K}}$*, a commutative Frobenius object is a commutative Frobenius algebra,* [3].

This leads to the following question: **Is there a classification of nTFTs for $n > 0$?**

In the above cases, the classification was given by a classification of n-dimensional manifolds with boundary obtained by decomposing the manifold into smaller, elementary pieces. For $n = 1, 2$, such a decomposition can be obtained by Morse theory, by "cutting" the bordism along the time axis given by a Morse function.

To implement a similar idea in higher dimensions, one"time axis" doesn't suffice to get a full classification of n-dimensional manifolds with boundary. Introducing several time axes, i.e. decomposing the manifold in different directions, such as by triangulating, might again make it possible to study our manifold from the decomposition into smaller, elementary pieces. However, now the decomposition might have corners and we need our TFT to include data attached to these lower dimensional pieces. This leads to "higher categories", informally speaking this means that we now also have maps between morphisms, which are called "2-morphisms", and perhaps maps between these, and so on.

Definition 1.6. *Informally speaking, the data of an* n-category *consists of*

- *objects*
- *1-morphisms between objects*
- *2-morphisms between 1-morphisms*
- $\vdots$
- *n-morphisms between $(n-1)$-morphisms,*

and coherent compositions which are associative in a suitable way.

We will see precise definitions later. In the case $n = 2$, you might have seen examples under the names of 2-categories (strictly associative) or bicategories (associative up to coherence).

In [4], a bicategory of n-cobordisms is defined. Without going into too many details, it is roughly the following.

Definition 1.7. *The bicategory* $n\mathrm{Cob}^{ext}$ *has*

- *(n-2)-dimensional smooth closed manifolds as objects,*
- *1-morphisms are $(n-1)$-dimensional 1-bordisms between objects, and*
- *2-morphisms are isomorphism classes of n-dimensional 2-bordisms between 1-morphisms,*

where

(1) *a* 1-bordism from an object Y_0 to an object Y_1 *is an $(n-1)$-dimensional 1-bordism, i.e. a smooth compact $(n-1)$-dimensional manifold with boundary W, together with a decomposition and isomorphism*

$$\partial W = \partial_{in} W \amalg \partial_{out} W \cong Y_0 \amalg Y_1;$$

(2) *for two 1-bordisms W_0 and W_1 from Y_0 to Y_1, a* 2-bordism from W_0 to W_1 *is an n-dimensional manifold with faces together with a decomposition of the boundary into faces $(\partial_0 S, \partial_1 S)$ such that $\partial_0 S \cap \partial_1 S$ again is a face, and equipped with*

- *a decomposition and isomorphism*

$$\partial_0 S = \partial_{0,in} S \amalg \partial_{0,out} S \xrightarrow{\sim} W_0 \amalg W_1,$$

- *a decomposition and isomorphism*

$$\partial_1 S = \partial_{1,in} S \amalg \partial_{1,out} S \xrightarrow{\sim} Y_0 \times [0,1] \amalg Y_1 \times [0,1].$$

(3) *Two 2-bordisms S, S' are* isomorphic *if there is a diffeomorphism $h : S \to S'$ compatible with the boundary data.*

Vertical and horizontal compositions of 2-morphisms are defined by choosing collars and gluing. This is well-defined because 2-morphisms are isomorphism classes of 2-bordisms, and thus the composition does not depend on the choice of the collar. However, composition of 1-morphisms requires the use of a choice of a collar, which requires the axiom of choice, and then composition is defined by the unique gluing. However, this gluing is associative only up to non-canonical isomorphism of 1-bordisms which gives a canonical isomorphism class of 2-bordisms realizing the associativity of horizontal composition in the axioms of a bicategory.

It is symmetric monoidal, with symmetric monoidal structure given by taking disjoint unions.

Definition 1.8. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal bicategory. A* 2-extended nTFT *valued in $\mathcal{C}^\otimes$ is a symmetric monoidal functor*

$$n\mathrm{Cob}^{ext} \longrightarrow \mathcal{C}^\otimes.$$

What is natural target for a TFT based on $2\mathrm{Cob}^{ext}$?

Answer: There is not a single one, but many choices! One possible choice: $\mathrm{Alg}^{bicat} = \mathrm{Alg}^{bicat}(\mathrm{V{\small ECT}}_{\mathbb{K}})$, whose objects are $\mathbb{K}$-algebras, 1-morphisms are bimodules, and 2-morphisms are intertwiners.

Theorem 1.9. *(Cobordism Hypothesis in dimension 2, oriented and unoriented case* [4]*) Let $\mathcal{C}^\otimes$ be a symmetric monoidal bicategory. There is an equivalence of bicategories between the bicategory of 2-extended oriented 2TFTs valued in $\mathcal{C}^\otimes$ and the 2-groupoid of "oriented-2-dualizable" objects in $\mathcal{C}^\otimes$ given by evaluation at a point. For $\mathcal{C}^\otimes = \mathrm{Alg}^{bicat}$, being "oriented-2-dualizable" unravels to being a separable symmetric Frobenius algebra.*

Theorem 1.10. *(Cobordism Hypothesis in dimension 2, framed case* [5]*) Let $\mathcal{C}^\otimes$ be a symmetric monoidal bicategory. There is an equivalence of bicategories between the bicategory of 2-extended framed 2TFTs valued in $\mathcal{C}^\otimes$ and the 2-groupoid of "2-dualizable" objects in $\mathcal{C}^\otimes$ given by evaluation at a point. For $\mathcal{C}^\otimes = \mathrm{Alg}^{bicat}$, for an algebra to be "2-dualizable" unravels to being separable, (finitely generated,) and projective as a $\mathbb{K}$-module (i.e. for* $\mathrm{char}\mathbb{K} = 0$, *finite dimensional semi-simple $\mathbb{K}$-algebra).*

Remark 1.11. A separable symmetric Frobenius algebra is the same as a 2-dualizable algebra with a non-degenerate symmetric trace.

For general n, it is very difficult to write down good models for weak n-categories, and for bordism categories, weakness is necessary. However, there is a notion of (∞, n)-categories using topology, for which there turns out to be a bordism (∞, n)-category Bord_n. We will study this in more detail later on.

Definition 1.12. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal (∞, n)-category. A* fully extended nTFT *is a symmetric monoidal functor*

$$\mathrm{Bord}_n \longrightarrow \mathcal{C}^\otimes.$$

Remark 1.13. Note that "fully extended" here means n-extended. Such a TFT often is called "fully local".

First formulated by Baez and Dolan in [6], the classification of fully extended TFTs goes by the name of the cobordism hypothesis:

Theorem 1.14. *(Cobordism Hypothesis after Hopkins-Lurie, Lurie [7]) There is an equivalence between the (∞, n)-category of fully extended framed nTFTs valued in a symmetric monoidal (∞, n)-category $\mathcal{C}^{\otimes}$ and the ∞-groupoid of "fully dualizable" objects in $\mathcal{C}^{\otimes}$ given by evaluation at a point.*

Remark 1.15. There is a new approach to the Cobordism Hypothesis by Francis-Ayala using factorization homology techniques [8].

Our goal for this week will be to construct an example of such a fully extended TFT! This will serve as the motivation to introduce some algebraic and categorical structures which are interesting in themselves and ubiquitous in mathematical physics and other fields in mathematics.

2. The E_n-operad

2.1. **Operads.** Operads were first introduced by May in [9] as a mathematical object encoding operations. (Fun fact: "Operad" is a portmanteau of the words "operation" and "monad", coined by May.) The main reference used here is [10].

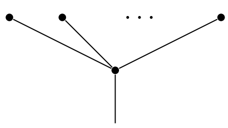
Definition 2.2. *A* (non-symmetric) operad of $\mathbb{K}$-vector spaces *is a family $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$, where $\mathcal{O}(n)$ is a $\mathbb{K}$-vector space together with*

- *an element $I \in \mathcal{O}(1)$ called the unit, and*
- *associative and unital composition maps*

$$m(n_1, \ldots, n_k) : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \longrightarrow \mathcal{O}(n_1 + \cdots + n_k).$$

Example 2.3. Let V be a $\mathbb{K}$-vector space. Then the collection $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$ is an operad with unit given by $\mathrm{id}_V \in \mathrm{Hom}(V, V)$. The composition maps $m(i_1, \ldots, i_k)$ are given by composition of endomorphisms.

Example 2.4. For $n \geq 1$, let $\mathrm{As}(n) = \mathbb{K}$ and $\mathrm{As}(0) = 0$. You can picture a basis for $\mathrm{As}(n)$ as trees with n inputs:



We can modify the above definition now setting $\mathrm{As}_+(n) = \mathbb{K}$. This is a unital version, as we will see later. In both cases, composition maps are given by multiplication of scalars $\mathbb{K} \otimes \cdots \otimes \mathbb{K} \to \mathbb{K}$.

Definition 2.5. *A morphism of operads $f : \mathcal{O} \to \mathcal{P}$ is a collection of maps $\{f(n) : \mathcal{O}(n) \to \mathcal{P}(n)\}_{n\geq 0}$ compatible with the units and composition maps, i.e.*

- *$f(1)(I_{\mathcal{O}}) = I_{\mathcal{P}}$, and*
- *$m_{\mathcal{P}}(n_1, \ldots, n_k) \circ f(k) \otimes f(n_1) \otimes \cdots \otimes f(n_k) = f(n_1 + \cdots + n_k) \circ m_{\mathcal{O}}(n_1, \ldots, n_k)$.*

Definition 2.6. *Let $\mathcal{O}$ be an operad. An* $\mathcal{O}$-algebra structure *on a $\mathbb{K}$-vector space V is a morphism of operads $\mathcal{O} \to \mathrm{End}_V$. A* morphism of $\mathcal{O}$-algebras *$g : V \to V'$ is a morphism of the underlying vector spaces which induces a commuting triangle*

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathrm{End}_V \\ & \searrow & \downarrow g \\ & & \mathrm{End}_{V'} \end{array}$$

The category of $\mathcal{O}$-algebras is denoted by $\mathrm{Alg}_{\mathcal{O}}$.

Remark 2.7. Unravelling the definition, an $\mathcal{O}$-algebra structure on V gives maps

$$\mathcal{O}(n) \otimes V^{\otimes n} \longrightarrow V$$

which are compatible with the units and composition maps. Moreover, given a morphism of operads $f : \mathcal{O} \to \mathcal{P}$, any $\mathcal{P}$-algebra structure on V can be pulled back along f to obtain an $\mathcal{O}$-algebra structure on V.

Example 2.8. (1) A As-algebra structure on V is the same as a associative algebra structure on V: denote the image of  by $\mu : V \otimes V \to V$. It is associative, since the compositions

give the same element in As(3) (since this is one-dimensional), and thus the images $\mu \circ (\mu \otimes \mathrm{id}_V)$ and $\mu \circ (\mathrm{id}_V \otimes \mu)$ must agree. Note that since $\mathrm{As}(1) = \mathbb{K}I$ and I is sent to $\mathrm{id}_V \in \mathrm{End}(V)$, and $\mathrm{As}(0) = 0$, these do not give extra structure.

(2) Similarly, in the unital case, an As_+-algebra structure on V is the same as a unital associative algebra structure on V.

(3) There is an operad Lie which governs Lie algebras.

2.9. **Variations.**

2.9.1. *Symmetric operads.* Symmetric operads allow to also have an operad governing commutative algebras:

Definition 2.10. *A* symmetric operad of $\mathbb{K}$-vector spaces *is a family* $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$*, where* $\mathcal{O}(n)$ *is a* $\mathbb{K}$*-vector spaces equipped with a right action of the symmetric group* S_n *on* n *elements together with*

- *an element* $I \in \mathcal{O}(1)$ *called the unit, and*
- *associative, unital, and equivariant composition maps*

$$m(n_1, \ldots, n_k) : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \longrightarrow \mathcal{O}(n_1 + \cdots + n_k).$$

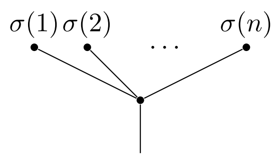
Example 2.11. End_V is a symmetric operad with the same definition as above, but additionally with an action of S_n given by permutation of the input elements:

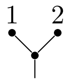
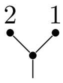
$$\sigma \cdot f(v_1, \ldots, v_n) = f(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}).$$

This allows to define algebras for an operad similarly to the non-symmetric case.

Example 2.12. For $n \geq 1$, let $\mathrm{Com}(n) = \mathbb{K}$ with trivial S_n-action, and $\mathrm{Com}(0) = 0$. A unital version is given by setting $\mathrm{Com}_+(n) = \mathbb{K}$ for all $n \geq 0$. Then Com-algebras are exactly the commutative algebras and Com_+-algebras are unital commutative algebras. Similarly to above, the image of the unique operation in Com(2) gives a binary operation, and the equivariance with respect to the S_2-action implies that it must be commutative.

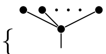
Example 2.13. Associative algebras allow to be encoded by a symmetric operad as well: For $n \geq 1$, let $\mathrm{Ass}(n) = \mathbb{K}[S_n]$, the regular representation of the symmetric group, and $\mathrm{Ass}(0) = 0$. You can picture a basis for $\mathrm{Ass}(n)$ as trees with n inputs labelled by $\sigma(i)$ for $\sigma \in S_n$:



Thus for $n = 2$ we have two elements,  and  . Thus, compared to the non-symmetric operads, we now have extra labellings.

2.13.1. *Other enrichments.* In the above definition, an operad was a collection of $\mathbb{K}$-vector spaces with extra structure. We can replace the choice of the category $\mathrm{VECT}_{\mathbb{K}}$ by any other symmetric monoidal category $\mathcal{S}$. Examples often encountered are the category of sets $\mathcal{S} = \mathrm{SET}$ *(set operads)*, the category of chain complexes $\mathcal{S} = \mathrm{CH}_{\mathbb{K}}$ *differential graded (=dg) operads*, and a category of spaces such as topological spaces $\mathcal{S} = \mathrm{TOP}$ *topological operads* or simplicial sets $\mathcal{S} = \mathrm{sSET}$ *simplicial operads.*

Example 2.14. For $\mathcal{S} = \text{Set}$, there is an analog of As (resp. Com and Ass) governing monoids,

$$\text{As}(n) = \{*\} = \{$$  $$\}$$

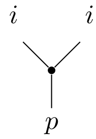
(respectively $\text{cMon}(n) = *$ and $\text{Mon}(n) = S_n$ in the symmetric case).

Exercise 2.15. For $\mathcal{S} = \text{Ch}_{\mathbb{K}}$, of course any operad over $\text{Vect}_{\mathbb{K}}$ also is one over $\text{Ch}_{\mathbb{K}}$ viewing a vector space as a chain complex concentrated in degree 0. However, now there is more flexibility, and leads to interesting problems when changing the underlying chain complex by a quasi-isomorphism, or more generally, by a homotopy of chain complexes:

(1) Let

$$h \circlearrowleft (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$
$$\text{id}_A - ip = d_A h + h d_A \ ,$$

be a homotopy datum of chain complexes, i.e. i and p are morphisms of chain complexes and h is a map of degree +1. If  $= m : A \otimes A \to A$ is an associative structure on (A, d_A), show that



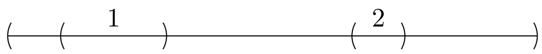
$$= p \circ m \circ i^{\otimes 2} : H \otimes H \longrightarrow H$$

in general is not associative.

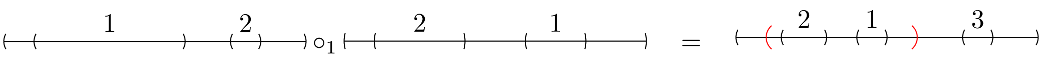
(2) Extract the dg operad governing the structure on (H, d_H). This is called the A_∞-*operad*.

(3) Given a homotopy datum of chain complexes as in (1), if (A, d_A) is an A_∞-algebra, is the transferred structure on (H, d_H) again an A_∞-structure?

Example 2.16. For $\mathcal{S} = \text{Top}$, consider the topological symmetric operad E_1, the *little intervals operad*, defined as follows. The space $\text{E}_1(n)$ is the space of rectilinear embeddings of n disjoint labelled intervals into $(0, 1)$, i.e. an element in $\text{E}_1(2)$ is given by n disjoint labelled subintervals of $(0, 1)$, e.g.



Composition is given by inserting the interval into one of the embedded ones and the action of the symmetric group is by permuting the labels of the balls.



Note that there is a morphism of operads

$$\text{E}_1 \longrightarrow \text{Mon},$$

which moreover level-wise is a homotopy equivalence.

Thus, any monoid (in spaces) in particular is an E_1-algebra in spaces. Moreover, we can apply the singular chain functor component-wise to obtain an operad E_1 in chain complexes. Then, the above map shows that any associative algebra is an E_1-algebra in chain complexes.

Another example of E_1-algebras is given by loop spaces: Let $(X, *)$ be a pointed topological space. Then $\Omega X = \text{Map}(S^1, X)$ is an E_1-algebra: since $S^1 = [0, 1]/(0 \sim 1)$, any $f \in \text{E}_1(n)$ determines a map $f : S^1 \to S^1 \amalg_* \ldots \amalg_* S^1$ by sending everything outside the intervals to the base point. Concatenation with f determines a map $\Omega X^{\times n} \to \Omega X$.

2.16.1. *Colored operads.* Just as an operad encoded operations on a single vector space, a colored operad encodes operations on a set of vector spaces indexed by a set of colors. Composition is only possible if the color of the output and the corresponding input match.

Definition 2.17. *Let* $\mathcal{S}^{\otimes}$ *be a symmetric monoidal category. A* colored (symmetric) $\mathcal{S}$-operad *or* (symmetric) $\mathcal{S}$-enriched multicategory $\mathcal{O}$ *consists of*

- *a set of* colors $\mathrm{Col}(\mathcal{O})$,
- *for any finite sequence* $(c_1, \dots, c_n; c)$ *of colors, an object* $\mathcal{O}(c_1, \dots, c_n; c) \in \mathcal{S}$,
- *for any* $n \geq 1$, *a right* S_n*-action: for* $\sigma \in S_n$,
$$\sigma^* : \mathcal{O}(c_1, \dots, c_n; c) \longrightarrow \mathcal{O}(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c),$$
- *for any color* $c \in \mathrm{Col}(\mathcal{O})$, *an element* $I_a \in \mathcal{O}(a; a)$ *called the unit,*
- *associative, unital, and equivariant composition maps*
$$\mathcal{O}(c_1, \dots, c_n; c) \otimes \mathcal{O}(d_1^1, \dots, d_{k_1}^1; c_1) \otimes \cdots \otimes \mathcal{O}(d_1^n, \dots, d_{k_n}^n; c_n) \longrightarrow \mathcal{O}(d_1^1, \dots, d_{k_n}^n; c).$$

The non-symmetric version is similar omitting the S_n*-action.*

Example 2.18. Let $V_1, \dots, V_n$ be $\mathbb{K}$-vector spaces. Let $\mathrm{End}_{V_1 \oplus \cdots V_n}$ be the colored operad with colors $\mathrm{Col}(\mathrm{End}_{V_1 \oplus \cdots \oplus V_n}) = \{V_1, \dots, V_n\} \cong \{1, \dots, n\}$, and
$$\mathrm{End}_{V_1 \oplus \cdots V_n}(V_{c_1}, \dots, V_{c_n}; V_c) = \mathrm{Hom}(V_{c_1} \otimes \cdots \otimes V_{c_n}, V_c).$$

Example 2.19. Let Bimod be the colored (non-symmetric) operad with 3 colors, L, M, R. We let
$$\mathrm{Bimod}(c_1, \dots, c_n; c) = \begin{cases} * & \text{if } c_1 = \cdots = c_n = c = L \\ * & \text{if } c_1 = \cdots = c_n = c = R \\ * & \text{if } c_1 = \cdots = c_{i-1} = L, c_i = c = M, \text{ and } c_{i+1} = \cdots = c_n = c = R \\ \emptyset & \text{otherwise.} \end{cases}$$

Example 2.20. A map $\mathbb{R} \to \mathbb{R}$ is called a rectilinear embedding if it has the form $x \mapsto ax + b$. Let $H = \mathbb{R}_{\geq 0}$. Consider the following colored topological operad E_1^{∂}. It has two colors $-, \bullet\!-$. We let
$$\mathrm{E}_1^{\partial}(c_1, \dots, c_n; c) = \begin{cases} \mathrm{Emb}^{rlin}(\mathbb{R}^{\amalg n}, \mathbb{R}) & \text{if } c_1 = \cdots = c_n = c = - \\ \mathrm{Emb}^{rlin}(\mathbb{R}^{\amalg n} \amalg H, H) & \text{if } c_1 = \cdots = c_{i-1} = -, \text{ and } c_i = c = \bullet\!-. \\ \emptyset & \text{otherwise.} \end{cases}$$

Exercise 2.21. Generalize the above example to an operad $\mathrm{E}_1^{-\bullet-}$ with three colors $-\!\bullet, -\!\bullet\!-, \bullet\!-$ which is to Bimod as E_1 is to Ass.

Example 2.22. Let X be a topological space. Let Disj_X be the colored operad with colors the open sets in X, $\mathrm{Col}(\mathrm{Disj}_X) = \mathrm{Open}(X)$, and
$$\mathrm{Disj}_X(U_1, \dots, U_n; V) = \begin{cases} * & \text{if } U_1 \amalg \dots \amalg U_n \subset V, \\ \emptyset & \text{else.} \end{cases}$$
This colored operad will play an important role in section 3.

Definition 2.23. *Let* $\mathcal{O}$ *be a colored* $\mathcal{S}$*-operad. An* $\mathcal{O}$-algebra *is an object in* $\mathcal{S}^{\mathrm{Col}(\mathcal{O})}$, *i.e. a map* $A : \mathrm{Col}(\mathcal{O}) \to \mathcal{S}$ *together with compatible morphisms*
$$\mathcal{O}(c_1, \dots, c_n; c) \otimes A(c_1) \otimes \cdots \otimes A(c_1) \longrightarrow A(c).$$
The category of $\mathcal{O}$*-algebras in* $\mathcal{S}$ *is denoted by* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S})$.

Construction 2.24. Assume that $\mathcal{S}$ has finite coproducts and these commute with $\otimes$ in each variable. Let $\mathcal{O}$ be a colored symmetric $\mathcal{S}$-operad. We can define the *free symmetric monoidal* $\mathcal{S}$*-enriched category* $\textsc{Env}(\mathcal{O})$ as follows:

- Objects are finite sequences of colors, which you should think of as finite formal tensor products of colors;
- Morphisms from $(c_1, \dots, c_k)$ to $(d_1, \dots, d_l)$ are elements in
$$\coprod_{\alpha \in \mathrm{Hom}_{\textsc{Set}}(\underline{k}, \underline{l})} \bigotimes_{j \in \underline{l}} \mathcal{O}((c_i)_{i \in \alpha^{-1}(j)}; d_j),$$
composition of morphisms arises from the composition maps;
- the symmetric monoidal structure is given by concatenation.

The assignment $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ has a right adjoint given by the following construction.

Construction 2.25. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $\mathcal{S}$-enriched category. Its *underlying colored operad* $\mathcal{U}(\mathcal{C})$ has

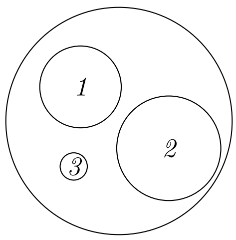
- colors are the objects of $\mathcal{C}$,
- $\mathcal{U}(\mathcal{C})(c_1, \ldots, c_k; c) = \mathcal{C}(\bigotimes_{i \in \underline{k}} c_i, c)$.

By adjunction, $\mathcal{O}$-algebras in $\mathcal{S}$ correspond precisely to symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to \mathcal{S}$. This leads to many more examples!

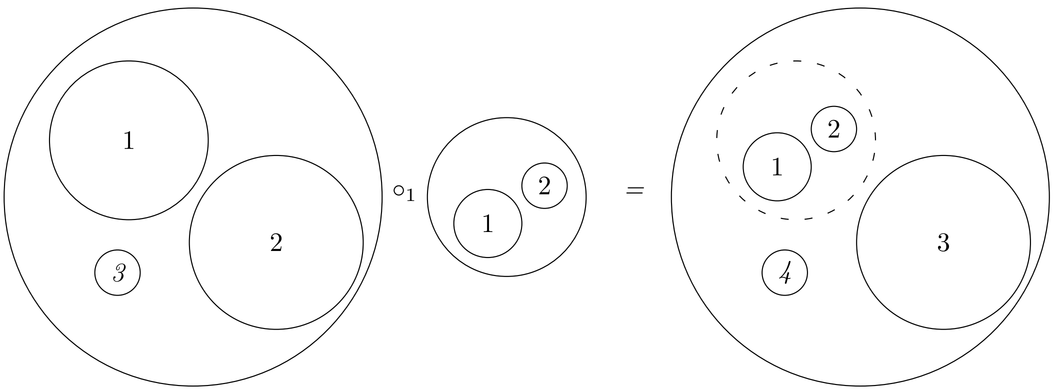
Example 2.26. A n-dimensional topological field theory with values in $\mathcal{S}^{\otimes}$ is a algebra for $\mathcal{U}(n\mathrm{Cob})$ in $\mathcal{S}^{\otimes}$.

2.27. **Several models for the E_n-operad.** We have seen the first incarnation of the E_1-operad as the little intervals operad in Example 2.16. Its generalization, the topological operad of *little k-disks*, was the first example studied by May which led to the definition of operads in the first place.

Definition 2.28. *The topological space $\mathrm{D}_k(n)$ is the space of n disjoint subdisks in the unit disk B^k in $\mathbb{R}^k$. Thus, an element in this space is a configuration of n pairwise disjoint k-dimensional subballs in B^k, determined by n continuous functions $S^{k-1} \to B^k$ whose images are disjoint. This description endows the set with a topology. For $k = 2$ and $n = 3$, an example of an element in $\mathrm{D}_2(3)$ is*



Composition is given by inserting a ball into one of the embedded balls and the action of the symmetric group is by permuting the labels of the balls.



This operad has several slight modifications:

Definition 2.29. *Choose a norm on $\mathbb{R}^k$ and let B^k be the unit ball for this norm. Let $C_k(n)$, $\square_k(n)$, and $\mathrm{Diff}_k^{fr}(n)$ be the spaces of embeddings $(B^k)^{\amalg n} \hookrightarrow B^k$ which have the following form when restricted to a single B^k:*

$C_k(n)$**:** $(x_1, \ldots, x_k) \mapsto (a_1 + bx_1, a_2 + bx_2, \ldots, a_k + bx_k)$ *for* $a_1, \ldots, a_k \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$,

$\square_k(n)$**:** $(x_1, \ldots, x_k) \mapsto (a_1 + b_1x_1, a_2 + b_2x_2, \ldots, a_k + b_kx_k)$ *for* $a_1, \ldots, a_k \in \mathbb{R}$, $b_1, \ldots, b_k \in \mathbb{R}_{>0}$,

$\mathrm{Diff}_k^{fr}(n)$**:** *framed embedding*

There are inclusions

$$C_k \longrightarrow \square_k \longrightarrow \mathrm{Diff}_k^{fr},$$

which in fact are level-wise homotopy equivalences.

Remark 2.30. Note that if we choose the standard norm on $\mathbb{R}^k$, C_k is the little k-disks operad. If we take the maximum norm on $\mathbb{R}^k$, C_k goes under the name of *little k-cubes operad*, and $\square_k$ is called the *little k-rectangles operad.*

Definition 2.31. *Any of the above equivalent operads is called an* E_k*-operad.*

Example 2.32. The motivating examples of E_k-algebras which led to the definition of the little k-disks operad are k-fold loop spaces generalizing the example of an E_1-algebra above: Let $(X, *)$ be a pointed topological space and consider $\Omega^k X = \mathrm{Map}(S^k, X)$. Now any $f \in \mathrm{E}_k(n)$ determines a map

$$S^k = \overline{B^k}/\partial\overline{B^k} \longrightarrow \overline{B^k}/(\overline{B^k} \setminus \mathrm{im} f) \longrightarrow S^k \coprod_{*} \cdots \coprod_{*} S^k.$$

In the category of topological spaces, in fact, it is due to May that this example is essentially the only one:

Theorem 2.33. *(May's recognition principle) Let Y be a connected topological space. If it has an E_k-algebra structure, it is homotopy equivalent to the k-fold loop space of some pointed topological space X,*

$$Y \sim \Omega^k(X).$$

There are maps of non-symmetric operads

$$\mathrm{E}_k \longrightarrow \mathrm{E}_{k+1} \longrightarrow \mathrm{Com},$$

where the first map is induced by the inclusion the $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}, (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k+1}, 0)$. In particular, any commutative algebra is an E_k-algebra for any k, and from any E_k-algebra we can extract an E_1-algebra, which is a homotopy associative algebra. Thus, we should think of E_k-algebras as interpolating between associative and commutative.

Example 2.34. For $k = 2$, we will see that, in the correct sense, braided monoidal categories are examples of E_2-algebras.

2.34.1. *Dunn's additivity.* Without going into technical details, there is a tensor product of operads, defined in certain contexts, designed to implement that

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{S})) \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{S}).$$

Using this tensor product, Dunn's additivity theorem, proven in full generality by Lurie in [11], states that there is an equivalence

$$\mathrm{E}_k \otimes \mathrm{E}_{k'} \simeq \mathrm{E}_{k+k'},$$

which has the important consequence that

$$\mathrm{Alg}_{\mathrm{E}_k}(\mathrm{Alg}_{\mathrm{E}_{k'}}(\mathcal{S})) \simeq \mathrm{Alg}_{\mathrm{E}_{k+k'}}(\mathcal{S}).$$

Even though explaining explaining this theorem would be beyond the scope of these lectures, let us try to understand some important consequences of this theorem.

First, roughly, the theorem says that an E_k-structure is the same as k homotopy associative structures which are compatible. This has interesting consequences as we see in the following lemma.

Lemma 2.35. *(Eckmann-Hilton trick) Consider a vector space V endowed with two binary unital operations*

$$\cdot, \circ : V \otimes V \longrightarrow V,$$

such that for every $a, b, c, d \in V$,

$$(a \cdot b) \circ (c \cdot d) = (a \circ c) \cdot (b \circ d).$$

Then the operations are equal, commutative, and associative.

Proof. Exercise: show that units are the same. Then consider the following square:

a	b
c	d

Read the horizontal dividing lines as "$\cdot$" and the vertical ones as "$\circ$". Then first reading the rows, then the columns gives the right hand side of the equation; first the columns, then the rows gives the left hand side of the equation. Then the squares

a	1
1	b

and

1	a
b	1

give the equations

$$a \circ b = a \cdot b \qquad \text{and} \qquad a \circ b = b \cdot a,$$

so the operations are equal and commutative. Associativity follows from the square

a	b
1	c

□

The consequence of this lemma is that any E_2-algebra in $\mathrm{V{\small ECT}}_{\mathbb{K}}$ is commutative! Thus, to get something non-trivial, we really need some "homotopy" or "higher"!

Lemma 2.36. *Endowing a category with the structure of an* E_2*-algebra is equivalent to endowing it with a braided monoidal structure.*

Proof. See [11], Example 5.1.2.4. □

3. Factorization algebras

Factorization algebras were first defined by Beilinson and Drinfeld in the algebraic setting ([12]) inspired by conformal field theory. Their topological incarnation encodes the structure of the observables of a perturbative quantum field theory, which is explained in [13].

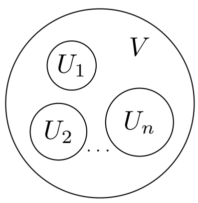
Let X be a topological space. Recall the colored operad Disj_X from Example 2.22.

Definition 3.1. *A* prefactorization algebra on X with values in a symmetric monoidal category $\mathcal{S}^{\otimes}$ *is a* Disj_X*-algebra in* $\mathcal{S}^{\otimes}$*.*

Unravelling the definition, a prefactorization algebra is a the data of a functor

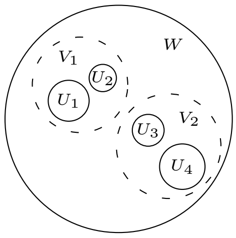
$$\mathcal{F} : \mathrm{Open}(X) \longrightarrow \mathcal{C}$$

together with for every $U_1 \amalg \ldots \amalg U_n \subseteq V$, we have a morphism



$$\rightsquigarrow \qquad f_{U_1,\ldots,U_n;V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V).$$

which if $n = 1$ is the morphism assigned by $\mathcal{F}$. These morphisms have to satisfy the following coherence condition: if $U_1 \amalg \cdots \amalg U_{n_i} \subseteq V_i$ and $V_1 \amalg \cdots \amalg V_k \subseteq W$, the following diagram commutes.



(for $k = n_1 = n_2 = 2$)

$$\rightsquigarrow \qquad \begin{array}{ccc} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) & \longrightarrow & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

Example 3.2. Let $(M, *)$ be a pointed topological space, X a topological space. Then

$$\begin{aligned} C^0_c(-, M) : \mathrm{Open}(X) &\longrightarrow (\textsc{Set}, \times) \\ U &\longmapsto C^0_c(U, M), \end{aligned}$$

where $C^0_c(U, M)$ are compactly supported continuous functions on U, is a prefactorization algebra. The structure maps are given by extending a function outside an open set by the chosen base point.

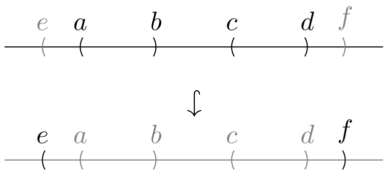
Example 3.3. (Algebras as factorization algebras) Let $X = \mathbb{R}$ and $\mathcal{S}^{\otimes} = \textsc{Vect}_{\mathbb{K}}$. Let A be a unital associative algebra over a field $\mathbb{K}$. Then the following assignment can be extended to a prefactorization algebra $\mathcal{F}_A$:

$$\mathcal{F}_A : (a_1, b_1) \amalg \cdots \amalg (a_n, b_n) \longmapsto A^{\otimes n}, \qquad \emptyset \longmapsto \mathbb{K}$$

and

$$\begin{array}{ccc} (a,b) & \overset{\mathcal{F}_A}{\longmapsto} & A \\ \big\downarrow & & \big\downarrow {\scriptstyle id_A} \\ (c,d) & \overset{\mathcal{F}_A}{\longmapsto} & A \end{array}$$

The morphisms $f_{U_1,\dots,U_n;V}$ are defined by multiplication in A, i.e. if $(a, b) \amalg (c, d) \subseteq (e, f)$ with $e < a < b < c < d < f$, the morphism $f_{(a,b),(c,d);(e,f)}$ is given by



$$\rightsquigarrow \quad \begin{array}{ccccc} \mathcal{F}_A((a,b)) & \otimes & \mathcal{F}_A((c,d)) & \longrightarrow & \mathcal{F}_A((e,f)) \\ \| & & \| & & \| \\ A & \otimes & A & \overset{m}{\longrightarrow} & A \end{array}$$

Associativity of m implies the coherence condition.

4. $(\infty, 1)$-categories – another higher flavor

Main references: [14], [15], [16]

Several models:

- Topologically/Simplicially enriched categories (cf. topological operads)
- complete Segal spaces (Rezk)

Examples: $\mathrm{Ch}_{\mathbb{K}}, \mathrm{Sp}$, every ordinary category

5. Factorization algebras revisited and factorization homology

Main references: [13], [17], [18], [19], [20]

- Descent in $(\infty, 1)$-category setting and algebraic examples
- factorization homology: definition as left Kan extension, generalized homology theory, gives locally constant factorization algebra

6. Factorization Homology as a fully extended TFT

Main references: [21], [22], [23], [24]

- complete n-fold Segal spaces
- $\mathrm{Bord}_n, \mathrm{Alg}_n$ as (∞, n)-categories
- sketch of the functor

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