DERIVED SYMPLECTIC GEOMETRY AND AKSZ TOPOLOGICAL FIELD THEORIES

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The main goal of this talk is to explain the construction of a family of "fully extended" topological field theories which describe semi-classical TFTs and use the language of derived symplectic geometry. I will first recall the necessary background upon which our construction is based.

Some derived symplectic geometry... The "spaces" studied in derived algebraic geometry are derived stacks, which add homotopical structure to the data of a scheme, or, more generally, a stack. One main reason for working in this setting is that (derived) intersection of derived stacks is well-behaved: in contrast to classical algebraic geometry, where it is crucial that the objects you want to intersect are transversal (e.g. for Bezout's Theorem), in the derived setting it always exists (up to homotopy) and captures the correct information.

In the seminal paper [PTVV13] the authors develop a theory of derived symplectic geometry in the context of derived algebraic geometry, which extends usual symplectic (algebraic) geometry. They introduce a notion of n-shifted symplectic structures on derived Artin stacks, which generalize ordinary symplectic structures on smooth varieties and schemes. Recall that classically, a symplectic structure provides an identification of the tangent and cotangent spaces. In the derived world, there now are tangent and cotangent *complexes*, and we ask for a quasi-isomorphism between them, possible with a degree shift n.

One example is that of the classifying stack BG for a reductive algebraic group. It is 2-shifted symplectic, with symplectic form given by a non-degenerate bilinear form.

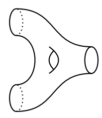
A large family examples comes from "transgression", a version of the AKSZ construction from [ASZK97] in the context of derived algebraic geometry: Given an n-shifted symplectic target Y and a suitable "d-oriented" source Σ, a pull-push construction along

$$\mathrm{Map}(\Sigma, Y) \xleftarrow{\;proj_2\;} \Sigma \times \mathrm{Map}(\Sigma, Y) \xrightarrow{\;ev\;} Y$$

constructs an $(n-d)$-shifted symplectic form on the mapping stack $\mathrm{Map}(\Sigma, Y)$.

In particular, this explains the existence of certain well-known symplectic structures on moduli spaces, such as the Atiyah-Bott symplectic form on the moduli space $\mathrm{Loc}_G(\Sigma) = \mathrm{Map}(\Sigma_B, BG)$ of G-local systems on a compact oriented surface.

...and some topological field theories. Recall [Ati88, Seg04] that an oriented topological field theory (TFT) is a symmetric monoidal functor out of the bordism category $d\mathrm{Cob}^{or}$ whose objects are closed $(d-1)$-dimensional manifolds and whose morphisms from M to N are d-dimensional cobordisms, i.e. manifolds Σ with boundary $\partial\Sigma \cong M \amalg N$,



$$\mathrm{M} \xrightarrow[\Sigma]{} \mathrm{N}\,.$$

Inspired by the work of Cattaneo-Mnev-Reshetikin [CMR14], Calaque showed in [Cal] that this construction extends to derived mapping stacks of surfaces with boundaries, and gives a structure which is a generalization of a Lagrangian: for the latter example above, let M be compact oriented 3-manifold M with boundary Σ. Then the inclusion

$$\mathrm{Loc}_G(M) \to \mathrm{Loc}_G(\Sigma)$$

has a Lagrangian structure.

Calaque shows that the above construction yields an TFT with values in $\mathrm{Lag}_1[n]$, whose objects are n-shifted symplectic stacks and whose morphisms from X to Y are *Lagrangian correspondences* from X to Y, i.e. Lagrangians $L \to \bar{X} \times Y$, where $\bar{X}$ is equipped with the opposite symplectic structure. That is, given an n-shifted symplectic stack Y, he constructs a symmetric monoidal functor

$$d\mathrm{Cob}^{or} \xrightarrow{\mathrm{Map}((-)_B, Y)} \mathrm{Lag}_1[n-d+1].$$

The main goal of joint work in progress with Damien Calaque and Rune Haugseng which I plan to outline in this talk is to extend this TFT to a *fully extended* one, that is, a symmetric monoidal functor out of a higher category (=(∞, d)-category) of cobordisms Bord_d^{or}, with target a higher category of Lagrangian correspondences $\mathrm{Lag}_d[n]$,

$$\mathrm{Bord}_d^{or} \xrightarrow{\mathrm{Map}((-)_B, Y)} \mathrm{Lag}_d[n].$$

In light of the Cobordism Hypothesis [BD95, Lur09], this is the fully extended TFT corresponding to Y, which is a fully dualizable object object in $\mathrm{Lag}_d[n]$.

As an $(\infty, 1)$-category, the natural target for the fully extended 1-dimensional TFT, $\mathrm{Lag}_1[n]$, has been constructed by Haugseng in [Hau14]. As a bicategory, it has been constructed in [AB16].

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